

$N = 2$ and $N = 4$ Supersymmetric mKdV and sinh-Gordon Hierarchies

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Abstract

Integrable models with higher $N = 2$ and $N = 4$ supersymmetries are formulated on reductions of twisted loop superalgebras $\widehat{sl}(2|2)$ and $\widehat{sl}(4|4)$ endowed with principal gradation. In case of the $\widehat{sl}(4|4)$ loop algebra a sequence of progressing reductions leads both to the $N = 4$ and $N = 2$ supersymmetric mKdV and sinh-Gordon equations.

The reduction scheme is induced by twisted automorphism and allows via dressing approach to associate to each symmetry flow of half-integer degree a supersymmetry transformation involving only local expressions in terms of the underlying fields.

1 Introduction

We further develop the systematic construction of supersymmetric integrable hierarchies in terms of the algebraic formalism proposed in reference [1]. In particular, we present a method leading to integrable models invariant under $N = 2$ and $N = 4$ extended supersymmetry realized in a local way (only expressions local in the underlying fields are involved). The method consists of the following key steps. The basic Lie algebraic objects in the scheme are the loop algebras $\widehat{sl}(2|2)$ and $\widehat{sl}(4|4)$ endowed with a principal gradation and a semisimple element E of degree one. The first step defines a symmetry structure of integrable models under construction. It involves dressing construction to associate symmetry transformations of $\widehat{sl}(2|2)$ and $\widehat{sl}(4|4)$ loop algebras to elements of the centralizer \mathcal{K} of E . In this scheme the isospectral deformations defining hierarchies of nonlinear flow equations correspond to the center of this centralizer. The rest of the centralizer becomes associated to (in general non-commuting) additional symmetry algebra of the underlying model. Especially, in models with principal gradation, which we are considering, the symmetry transformations corresponding to elements of the centralizer with the half-integer grades give raise to supersymmetry transformations. The identity element belongs to the $\widehat{sl}(2|2)$ and $\widehat{sl}(4|4)$ loop

algebras and that has a consequence of doubling number of the supersymmetry generators since squares of supersymmetry generators can be considered equivalent as long as they differ by identity element only. In general, graded loop algebras with principal gradation lead to non-local expressions for the supersymmetry transformations (e.g. the $N = 2$ super mKdV derived in [1] from the $\widehat{sl}(2|1)$ algebra). The non-locality can be traced back to a presence of a kernel term in the Lax operator required for consistency in general case. It is shown here that such term can be removed by a very judicious choice of an underlying relevant subalgebra within the loop structure. The key property of our reduction scheme is that it leads to reduced loop subalgebra with image of E containing only Cartan generators among even grade elements. The emerging abelian structure ensures that the supersymmetry transformations remain local thus allowing us to circumvent problem with non-local realization of supersymmetry.

We employ this framework to construct equations of motion for the generalized mKdV hierarchies invariant under supersymmetry without non-local terms. In particular, we derive the equations of motion for the $N = 2$ supersymmetric mKdV and sinh-Gordon models and show that they correspond to positive and negative grade time evolutions respectively and therefore to the same hierarchy.

Construction of $N = 4$ models involves two subsequent reductions. The first one, reduces the algebra $\widehat{sl}(4|4)$ into a semidirect product of $\widehat{U}(1)$ and $\widehat{sl}(2|2) \otimes \widehat{sl}(2|2)$. In the second reduction the relevant subalgebra is obtained by independently reducing each $\widehat{sl}(2|2)$ within its loop structure.

We were inspired by a number of papers devoted to a supersymmetric generalization of the Drinfeld-Sokolov construction. Inami and Kanno [2, 3] were first to consider a class of affine superalgebras with fermionic simple roots with the principal gradation. Delduc and Gallot [4] and soon after Madsen and Miramontes [5] realized importance of fermionic elements in the kernel which square to E for construction of a supersymmetric integrable hierarchy of the Drinfeld-Sokolov type.

Another set of related developments involves presence of non-local charges. Kersten [6] and Dargis and Mathieu [7, 8] obtained an infinite sequence of non-local (bosonic and fermionic) flow equations and conserved quantities in the framework of supersymmetric KdV equation (see also [9]). These flows emerge in algebraic formalism from (fermionic or bosonic) elements in \mathcal{K} with non-negative grade as it was also observed in [5] and later in [1].

There exists an extensive literature (see e.g. [10, 11, 12, 13, 14, 15, 16, 17]) devoted to construction of integrable models with extended supersymmetry based on a superspace. Such attempts use as a rule covariant derivatives in expressions for the Lax operator to ensure the supersymmetry invariance. In our attempt supersymmetry emerges from structure of symmetry transformations induced by fermionic directions in the underlying superalgebra.

The paper is organized as follows. In Section 2, the main features of the dressing approach are described in a way which sets a scene for supersymmetry invariant formulation of integrable hierarchies of evolution flows. Section 3 obtains $N = 2$ mKdV and sinh-Gordon equations as zero-curvature conditions based on a subalgebra of $\widehat{sl}(2|2)$. Extension of the above reduction process is applied in Section 4 to the $\widehat{sl}(4|4)$ algebra resulting in $N = 4$ mKdV and sinh-Gordon equations. The $N = 2$ mKdV and sinh-Gordon equations of Section 3 can be reproduced by setting some of the fields of $N = 4$ models to zero. We offer

some concluding remarks in Section 6. Technical details of $\widehat{sl}(4|4)$ algebra are relocated to Appendix A.

2 Dressing Formalism

In this Section we give a brief account of the dressing approach to the integrable models.

We will be working with a loop algebra $\widehat{\mathcal{G}}$ endowed with the principal gradation defined by a grading operator Q to be given below for each specific model. The gradation induces decomposition into graded subspaces $\widehat{\mathcal{G}} = \oplus_{n \in \mathbb{Z}} \widehat{\mathcal{G}}_n$ with $\widehat{\mathcal{G}}_n$ such that $[Q, \widehat{\mathcal{G}}_n] = n \widehat{\mathcal{G}}_n$.

Another fundamental object in this setting is a semisimple element E of grade one. The Kernel \mathcal{K} of E is a subalgebra of all elements commuting with E . Elements of \mathcal{K} generate algebra of symmetries commuting with isospectral deformations [18]. In the constructed models the supersymmetry transformations belong to \mathcal{K} and carry the half-integer grading.

Recall, that the center of kernel \mathcal{K} is defined as $C(\mathcal{K}) = \{x \in \mathcal{K} \mid [x, y] = 0 \ \forall y \in \mathcal{K}\}$. Elements of $C(\mathcal{K})$ of grade $n \in \mathbb{Z}$ are denoted as $E^{(n)}$ and in this notation $E \equiv E^{(1)}$. E induces decomposition $\widehat{\mathcal{G}} = \mathcal{K} \oplus \mathcal{M}$ where \mathcal{M} is an image of the adjoint operation $\text{ad}(E)X = [E, X]$.

To every positive grade element K_i in \mathcal{K} one associates a transformation δ_{K_i} through a map :

$$K_i \in \mathcal{K} \longrightarrow \delta_{K_i} \Theta = (\Theta K_i \Theta^{-1})_- \Theta, \quad (2.1)$$

where $(X)_-$ is a projection of $X \in \widehat{\mathcal{G}}$ on a strictly negative part of X in $\widehat{\mathcal{G}}_{<} = \oplus_{n=-1}^{-\infty} \widehat{\mathcal{G}}_n$ and Θ , the dressing matrix, is an exponential in $\mathcal{G}_{<}$:

$$\Theta = \exp \left(\sum_{i < 0} \theta^{(i)} \right) = \exp (\theta^{(-1)} + \theta^{(-2)} + \dots) . \quad (2.2)$$

The map $K_i \rightarrow \delta_{K_i}$ is a homomorphism [18]:

$$[\delta_{K_i}, \delta_{K_j}] \Theta = \delta_{[K_i, K_j]} \Theta. \quad (2.3)$$

It allows us to identify \mathcal{K} with algebra of symmetry transformations.

For $E^{(n)} \in C(\mathcal{K})$, the corresponding flows $\delta_{E^{(n)}}$ define through the map (2.1) the isospectral deformations of the model. They obviously commute among themselves and we denote them as partial derivatives $\partial/\partial t_n$. By definition :

$$\frac{\partial}{\partial t_n} \Theta(t) = (\Theta E^{(n)} \Theta^{-1})_- \Theta(t). \quad (2.4)$$

The Lax operator L is obtained by dressing the isospectral flow with $n = 1$ in equation (2.4). Let us identify t_1 with the space variable x . Then :

$$\begin{aligned} \partial_x(\Theta) &= (\Theta E \Theta^{-1})_- \Theta = [\Theta E \Theta^{-1} - (\Theta E \Theta^{-1})_+] \Theta \\ &= \Theta E - (E + [\theta^{(-1)}, E]) \Theta \\ &= \Theta E - (E + A_0) \Theta \end{aligned} \quad (2.5)$$

where $(\cdot)_+$ denotes projection on a positive subalgebra $\widehat{\mathcal{G}}_{\geq} = \oplus_{n=0}^{\infty} \widehat{\mathcal{G}}_n$. Note that $A_0 = [\theta^{(-1)}, E]$ is clearly in \mathcal{M} and of grade zero. This leads to the dressing expression for the Lax operator L :

$$\Theta (\partial_x + E) \Theta^{-1} = \partial_x + E + A_0 = L. \quad (2.6)$$

Similarly, for higher flows we obtain

$$\Theta \left(\frac{\partial}{\partial t_n} + E^{(n)} \right) \Theta^{-1} = \frac{\partial}{\partial t_n} + E^{(n)} + \sum_{i=0}^{n-1} D_n^{(i)} \quad (2.7)$$

with coefficients $D_n^{(i)}$ which can be found from identities:

$$(\Theta E^{(n)} \Theta^{-1})_+ = E^{(n)} + \sum_{i=0}^{n-1} D_n^{(i)}.$$

These dressing relations give rise to the zero-curvature conditions

$$\left[\partial_x + E + A_0, \frac{\partial}{\partial t_n} + E^{(n)} + \sum_{i=0}^{n-1} D_n^{(i)} \right] = \Theta \left[\partial_x + E, \frac{\partial}{\partial t_n} + E^{(n)} \right] \Theta^{-1} = 0. \quad (2.8)$$

The structure of the Lax operator changes when terms with the half-integer grades appear in $\widehat{\mathcal{G}} = \oplus_{n \in \mathbb{Z}} \widehat{\mathcal{G}}_{n/2}$. With these terms being present in the exponent of the dressing matrix equation (2.2) generalizes to :

$$\Theta = \exp \left(\sum_{i < 0} \theta^{(i)} \right) = \exp \left(\theta^{(-1/2)} + \theta^{(-1)} + \theta^{(-3/2)} + \dots \right) \quad (2.9)$$

and the form of the Lax operator obtained by the dressing procedure changes as follows :

$$\begin{aligned} \frac{\partial}{\partial t_1}(\Theta) &= (\Theta E \Theta^{-1})_- \Theta = [\Theta E \Theta^{-1} - (\Theta E \Theta^{-1})_+] \Theta \\ &= \Theta E + \left(E + [\theta^{(-1)}, E] + [\theta^{(-1/2)}, E] + \frac{1}{2} [\theta^{(-1/2)}, [\theta^{(-1/2)}, E]] \right) \Theta \\ &= \Theta E + (E + A_0 + A_{1/2} + k_0) \Theta. \end{aligned} \quad (2.10)$$

Here

$$A_0 = [\theta^{(-1)}, E] + \frac{1}{2} [\theta^{(-1/2)}, [\theta^{(-1/2)}, E]] \Big|_{\mathcal{M}} \in \mathcal{M} \quad (2.11)$$

$$A_{1/2} = [\theta^{(-1/2)}, E] \in \mathcal{M} \quad (2.12)$$

$$k_0 = \frac{1}{2} [\theta^{(-1/2)}, [\theta^{(-1/2)}, E]] \Big|_{\mathcal{K}} \in \mathcal{K} \quad (2.13)$$

where $\Big|_{\mathcal{K}}$ and $\Big|_{\mathcal{M}}$ denote projections on the kernel \mathcal{K} and image \mathcal{M} , respectively. This shows that, in case of a half-integer grading, a general expression for the Lax operator is

$$\mathcal{L} = \partial_x + E + A_0 + A_{1/2} + k_0. \quad (2.14)$$

The unconventional grade zero term k_0 residing in \mathcal{K} appears here due to the half-integer grading (encountered in case of $\widehat{sl}(n|m)$ algebras with principal gradation).

The presence of k_0 in the Lax operator L presents a problem for our formalism as it causes non-locality in the supersymmetry transformations [1] as follows from the zero-curvature equations we will describe below. A remedy employed in this paper is to work with subalgebras obtained by reductions in which the term $[\theta^{(-1/2)}, [\theta^{(-1/2)}, E]]$ vanishes automatically. This paper describes such construction for $\widehat{sl}(n|n)$ with $n = 2, 4$.

Consider now the case of a kernel \mathcal{K} which contains a constant grade one-half element $D^{(1/2)}$. According to (2.1) this term gives rise to the symmetry flow

$$\partial_{1/2}\Theta \equiv \delta_{D^{(1/2)}}\Theta = (\Theta D^{(1/2)}\Theta^{-1})_- \Theta. \quad (2.15)$$

As shown in [1], $\partial_{1/2}$ -flow enters the zero-curvature equation :

$$[\partial_x + E + A_0 + A_{1/2}, \partial_{1/2} + D^{(0)} + D^{(1/2)}] = 0, \quad (2.16)$$

where

$$D^{(0)} = [\theta^{(-1/2)}, D^{(1/2)}].$$

In presence of the half-integer grading the zero-curvature condition (2.8) gives way to (for reductions with $k_0 = 0$) :

$$\left[\partial_x + E + A_0 + A_{1/2}, \frac{\partial}{\partial t_n} + E^{(n)} + \sum_{k=1}^n (D_n^{(k)} + D_n^{(k-1/2)}) \right] = 0 \quad (2.17)$$

As described in [1] this zero-curvature approach extends to the negative flows $\partial/\partial t_{-n}$. The following three Lax operators :

$$\mathcal{D}_{+1} = \mathcal{L} = \partial_x + A_0 + A_{1/2} + E \quad (2.18)$$

$$\mathcal{D}_{+1/2} = \partial_{1/2} + D^{(0)} + D^{(1/2)} \quad (2.19)$$

$$\mathcal{D}_{-1} = \partial_{-1} + B j_{-1/2} B^{-1} + B E^{(-1)} B^{-1} \quad (2.20)$$

appear prominently in the dressing analysis. Here B is a non-singular matrix of grade zero and $j_{-1/2}$ is an element in \mathcal{M} of grade $-1/2$. The flows $\partial_{1/2}, \partial_{-1} = \partial/\partial t_{-1}$ and $\partial_x = \partial/\partial t_1$ are all commuting among themselves, being associated to $D^{(1/2)}, E^{(-1)}$ and E , respectively. By standard dressing argument this commutativity ensures the zero-curvature equations :

$$[\mathcal{D}_{+1/2}, \mathcal{D}_{-1}] = 0 \quad (2.21)$$

$$[\mathcal{D}_{+1/2}, \mathcal{D}_{+1}] = 0 \quad (2.22)$$

$$[\mathcal{D}_{+1}, \mathcal{D}_{-1}] = 0. \quad (2.23)$$

The brackets with $\mathcal{D}_{+1/2}$ act as compatibility equations which define supersymmetry transformations and ensure invariance of equations of motion ((2.23)), i.e.

$$[\partial_x + E + A_0 + A_{1/2}, \partial_{-1} + B j_{-1/2} B^{-1} + B E^{(-1)} B^{-1}] = 0 \quad (2.24)$$

under the supersymmetry transformation. The -1 grade component of the zero-curvature relation (2.24)

$$\partial_x (BE^{(-1)}B^{-1}) + [A_0, BE^{(-1)}B^{-1}] = 0 \quad (2.25)$$

is automatically satisfied for

$$A_0 = -\partial_x B B^{-1}. \quad (2.26)$$

Equation (2.21) is explicitly given by

$$[\partial_{1/2} + D^{(0)} + D^{(1/2)}, \partial_{-1} + B J_{-1/2} B^{-1} + BE^{(-1)}B^{-1}] = 0.$$

Its -1 grade component is equal to

$$\partial_{1/2} (BE^{(-1)}B^{-1}) + [D^{(0)}, BE^{(-1)}B^{-1}] = 0.$$

which has the obvious solution :

$$D^{(0)} = -\partial_{1/2} B B^{-1}. \quad (2.27)$$

With solutions (2.26) and (2.27) we can rewrite \mathcal{D}_{+1} and $\mathcal{D}_{+1/2}$ as

$$\mathcal{D}_{+1} = \mathcal{L} = \partial_x - \partial_x B B^{-1} + A_{1/2} + E \quad (2.28)$$

$$\mathcal{D}_{+1/2} = \partial_{1/2} - \partial_{1/2} B B^{-1} + D^{(1/2)}. \quad (2.29)$$

We now write explicitly all the zero-curvature equations in components. Equation (2.21) implies

$$\partial_{1/2} J_{-1/2} = [E^{(-1)}, B^{-1} D^{(1/2)} B] \quad (2.30)$$

$$\partial_{-1} (\partial_{1/2} B B^{-1}) = [B J_{-1/2} B^{-1}, D^{(1/2)}]. \quad (2.31)$$

From (2.22) we derive

$$[E, \partial_{1/2} B B^{-1}] = [A_{1/2}, D^{(1/2)}] \quad (2.32)$$

$$\partial_{1/2} A_{1/2} = [\partial_{1/2} B B^{-1}, A_{1/2}] - [\partial_x B B^{-1}, D^{(1/2)}]. \quad (2.33)$$

Finally, equation (2.23) yields

$$\partial_x J_{-1/2} = [E^{(-1)}, B^{-1} A_{1/2} B] \quad (2.34)$$

$$\partial_{-1} A_{1/2} = [E, B J_{-1/2} B^{-1}] \quad (2.35)$$

$$\partial_{-1} (\partial_x B B^{-1}) = [BE^{(-1)}B^{-1}, E] + [B J_{-1/2} B^{-1}, A_{+1/2}]. \quad (2.36)$$

The zero curvature condition

$$[\mathcal{D}_{1/2}, \mathcal{D}_n] = 0$$

for the higher flows generators

$$\mathcal{D}_n = \frac{\partial}{\partial t_n} + E^{(n)} + \sum_{k=1}^n (D_n^{(k-1)} + D_n^{(k-1/2)}),$$

ensures invariance of higher flows under the supersymmetry transformation.

3 Reduction of $\widehat{sl}(2|2)$ Algebra and $N = 2$ mKdV Supersymmetric System

Let us define basic objects. The principal grading for the $\widehat{sl}(2|2)$ algebra is defined in terms of the operator

$$Q = \lambda \frac{d}{d\lambda} + \frac{1}{2}(\alpha_1 + \alpha_3) \cdot H = \lambda \frac{d}{d\lambda} + \frac{1}{2} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]. \quad (3.1)$$

Grade one semisimple element E is chosen as

$$E = (E_{\alpha_1}^{(0)} + E_{-\alpha_1}^{(2)}) + (E_{\alpha_3}^{(2)} + E_{-\alpha_3}^{(0)}) + I^{(1)} = \left[\begin{array}{cc|cc} \lambda & 1 & 0 & 0 \\ \lambda^2 & \lambda & 0 & 0 \\ \hline 0 & 0 & \lambda & \lambda^2 \\ 0 & 0 & 1 & \lambda \end{array} \right]. \quad (3.2)$$

The odd (fermionic) part of the kernel of E consists of

$$\mathcal{K}_f = \{f_i^{(n+\frac{1}{2})}, i = 1, \dots, 4, \quad n \in \mathbb{Z}\} \quad (3.3)$$

with

$$\begin{aligned} f_1^{(n+\frac{1}{2})} &= (E_{\alpha_1+\alpha_2}^{(n-\frac{1}{2})} + E_{-\alpha_1-\alpha_2}^{(n+\frac{3}{2})}) + (E_{\alpha_2+\alpha_3}^{(n+\frac{3}{2})} + E_{-\alpha_2-\alpha_3}^{(n-\frac{1}{2})}), \\ f_2^{(n+\frac{1}{2})} &= (-E_{\alpha_1+\alpha_2}^{(n-\frac{1}{2})} + E_{-\alpha_1-\alpha_2}^{(n+\frac{3}{2})}) + (-E_{\alpha_2+\alpha_3}^{(n+\frac{3}{2})} + E_{-\alpha_2-\alpha_3}^{(n-\frac{1}{2})}), \\ f_3^{(n+\frac{1}{2})} &= (E_{\alpha_1+\alpha_2+\alpha_3}^{(n+\frac{1}{2})} + E_{-\alpha_1-\alpha_2-\alpha_3}^{(n+\frac{1}{2})}) + (E_{\alpha_2}^{(n+\frac{1}{2})} + E_{-\alpha_2}^{(n+\frac{1}{2})}), \\ f_4^{(n+\frac{1}{2})} &= (-E_{\alpha_1+\alpha_2+\alpha_3}^{(n+\frac{1}{2})} + E_{-\alpha_1-\alpha_2-\alpha_3}^{(n+\frac{1}{2})}) + (-E_{\alpha_2}^{(n+\frac{1}{2})} + E_{-\alpha_2}^{(n+\frac{1}{2})}), \end{aligned} \quad (3.4)$$

while the bosonic part $\mathcal{K}_b = \{K_i^{(n)}, i = 1, \dots, 3, \quad n \in \mathbb{Z}\}$ of the kernel \mathcal{K} contains

$$K_1^{(n)} = E_{\alpha_1}^{(n-1)} + E_{-\alpha_1}^{(n+1)}, \quad K_2^{(n)} = E_{\alpha_3}^{(n+1)} + E_{-\alpha_3}^{(n-1)}, \quad K_3^{(n)} = \lambda^n I.$$

where I is an identity matrix, traceless with respect to the supertrace.

Note, that in terms of generators from \mathcal{K}_b we can write

$$E = K_1^{(1)} + K_2^{(1)} + K_3^{(1)}.$$

Because K_3 commutes with the rest of the algebra the symmetry transformation δ_{K_3} defined according to (2.1) vanishes. Thus, the flows generated by, i.e.

$$\widetilde{E}^{(n)} = K_1^{(n)} + K_2^{(n)} - K_3^{(n)}$$

will give rise via relation (2.1) to flows identical to the isospectral deformations generated by $E^{(n)}$. As far as symmetry transformations go we can therefore identify $E^{(n)}$ with $\tilde{E}^{(n)}$. As we will see below this observation is crucial for extending supersymmetry by incorporating as supersymmetry flows the flows generated by elements of \mathcal{K}_f which square to E as well as those generated by elements which square to $\tilde{E}^{(1)}$.

We change now the basis in \mathcal{K}_f to :

$$\begin{aligned} F_1^{(n+\frac{1}{2})} &= \frac{1}{\sqrt{2}} \left(f_1^{(n+\frac{1}{2})} + f_3^{(n+\frac{1}{2})} \right), & F_2^{(n+\frac{1}{2})} &= \frac{1}{\sqrt{2}} \left(f_2^{(n+\frac{1}{2})} + f_4^{(n+\frac{1}{2})} \right), \\ F_3^{(n+\frac{1}{2})} &= \frac{1}{\sqrt{2}} \left(f_1^{(n+\frac{1}{2})} - f_3^{(n+\frac{1}{2})} \right), & F_4^{(n+\frac{1}{2})} &= \frac{1}{\sqrt{2}} \left(f_2^{(n+\frac{1}{2})} - f_4^{(n+\frac{1}{2})} \right). \end{aligned}$$

The advantage of using this basis is that its elements generate fermionic flows that square to isospectral deformations generated by $E^{(n)}$ (or $\tilde{E}^{(n)}$) according to anti-commutation relations of $F_i^{(n+\frac{1}{2})}$, $i = 1, \dots, 4$:

$$\{F_i^{(m+\frac{1}{2})}, F_i^{(n+\frac{1}{2})}\} = 2(-1)^{i+1} E^{(m+n+1)}, \quad i = 1, 2 \quad (3.5)$$

$$\{F_i^{(m+\frac{1}{2})}, F_i^{(n+\frac{1}{2})}\} = 2(-1)^i \tilde{E}^{(m+n+1)}, \quad i = 3, 4 \quad (3.6)$$

The remaining anti-commutation relations of $F_i^{(n+\frac{1}{2})}$, $i = 1, \dots, 4$ vanish. In this sense $F_i^{(\frac{1}{2})}$, $i = 1, \dots, 4$ generate the supersymmetry flows.

The fermionic part of the image \mathcal{M} of E consists of

$$\mathcal{M}_f = \{g_i^{(n+\frac{1}{2})}, i = 1, \dots, 4, \quad n \in \mathbb{Z}\} \quad (3.7)$$

where

$$\begin{aligned} g_1^{(n+\frac{1}{2})} &= (E_{\alpha_1+\alpha_2}^{(n-\frac{1}{2})} + E_{-\alpha_1-\alpha_2}^{(n+\frac{3}{2})}) - (E_{\alpha_2+\alpha_3}^{(n+\frac{3}{2})} + E_{-\alpha_2-\alpha_3}^{(n-\frac{1}{2})}), \\ g_2^{(n+\frac{1}{2})} &= (-E_{\alpha_1+\alpha_2}^{(n-\frac{1}{2})} + E_{-\alpha_1-\alpha_2}^{(n+\frac{3}{2})}) + (E_{\alpha_2+\alpha_3}^{(n+\frac{3}{2})} - E_{-\alpha_2-\alpha_3}^{(n-\frac{1}{2})}), \\ g_3^{(n+\frac{1}{2})} &= (E_{\alpha_1+\alpha_2+\alpha_3}^{(n+\frac{1}{2})} + E_{-\alpha_1-\alpha_2-\alpha_3}^{(n+\frac{1}{2})}) - (E_{\alpha_2}^{(n+\frac{1}{2})} + E_{-\alpha_2}^{(n+\frac{1}{2})}), \\ g_4^{(n+\frac{1}{2})} &= (-E_{\alpha_1+\alpha_2+\alpha_3}^{(n+\frac{1}{2})} + E_{-\alpha_1-\alpha_2-\alpha_3}^{(n+\frac{1}{2})}) + (E_{\alpha_2}^{(n+\frac{1}{2})} - E_{-\alpha_2}^{(n+\frac{1}{2})}), \end{aligned} \quad (3.8)$$

Again we will rather work with a different basis of \mathcal{M}_f given by

$$\begin{aligned} G_1^{(n+\frac{1}{2})} &= \frac{1}{\sqrt{2}} \left(g_1^{(n+\frac{1}{2})} + g_3^{(n+\frac{1}{2})} \right), & G_2^{(n+\frac{1}{2})} &= \frac{1}{\sqrt{2}} \left(g_2^{(n+\frac{1}{2})} + g_4^{(n+\frac{1}{2})} \right), \\ G_3^{(n+\frac{1}{2})} &= \frac{1}{\sqrt{2}} \left(g_1^{(n+\frac{1}{2})} - g_3^{(n+\frac{1}{2})} \right), & G_4^{(n+\frac{1}{2})} &= \frac{1}{\sqrt{2}} \left(g_2^{(n+\frac{1}{2})} - g_4^{(n+\frac{1}{2})} \right). \end{aligned} \quad (3.9)$$

There are four bosonic generators

$$\begin{aligned} M_1^{(n)} &= \alpha_1 \cdot H^{(n)}, & M_2^{(n)} &= -E_{\alpha_1}^{(n-1)} + E_{-\alpha_1}^{(n+1)}, \\ M_3^{(n)} &= \alpha_3 \cdot H^{(n)}, & M_4^{(n)} &= -E_{\alpha_3}^{(n+1)} + E_{-\alpha_3}^{(n-1)}, \end{aligned}$$

of the image \mathcal{M} of E . Note, that $M_1^{(n)}$ and $M_3^{(n)}$ are in the Cartan subalgebra. All four bosonic generators of \mathcal{M} are reproduced by the following anti-commutations relations between fermionic members of \mathcal{K} and \mathcal{M} :

$$\begin{aligned}\{G_1^{(m+\frac{1}{2})}, F_i^{(n+\frac{1}{2})}\} &= 2(-1)^{i+1}M_i^{(m+n+1)}, \quad i = 1, 2, 3, 4 \\ \{G_4^{(m+\frac{1}{2})}, F_i^{(n+\frac{1}{2})}\} &= -2M_{5-i}^{(m+n+1)}, \quad i = 1, 2, 3, 4.\end{aligned}$$

Using relations

$$\begin{aligned}\left[E^{(m)}, G_1^{(n+\frac{1}{2})}\right] &= 2G_2^{(m+n+\frac{1}{2})}, & \left[E^{(m)}, G_2^{(n+\frac{1}{2})}\right] &= 2G_1^{(m+n+\frac{1}{2})} \\ \left[E^{(m)}, G_3^{(n+\frac{1}{2})}\right] &= -2G_4^{(m+n+\frac{1}{2})}, & \left[E^{(m)}, G_4^{(n+\frac{1}{2})}\right] &= -2G_3^{(m+n+\frac{1}{2})}\end{aligned}$$

and

$$\begin{aligned}\left[E^{(m)}, M_1^{(n)}\right] &= 2M_2^{(m+n)}, & \left[E^{(m)}, M_2^{(n)}\right] &= 2M_1^{(m+n)} \\ \left[E^{(m)}, M_3^{(n)}\right] &= 2M_4^{(m+n)}, & \left[E^{(m)}, M_4^{(n)}\right] &= 2M_3^{(m+n)}\end{aligned}$$

one can obtain the remaining anti-commutation relations between $G_2^{(m+\frac{1}{2})}$, $G_3^{(m+\frac{1}{2})}$ and $F_i^{(n+\frac{1}{2})}$.

3.1 Reduction of $\widehat{sl}(2|2)$ and N=2 mKdV and Sinh-Gordon equations

The $\widehat{sl}(2|2)$ algebra splits in two disjoint sets of generators :

$$\begin{aligned}M_1^{(2n)}, M_3^{(2n)}, \quad M_2^{(2n+1)}, M_4^{(2n+1)}, \quad K_1^{(2n+1)}, K_2^{(2n+1)}, I^{(2n+1)} \\ G_1^{(2n+\frac{1}{2})}, G_3^{(2n+\frac{1}{2})}, F_2^{(2n+\frac{1}{2})}, F_4^{(2n+\frac{1}{2})}, \\ G_2^{(2n+\frac{3}{2})}, G_4^{(2n+\frac{3}{2})}, \quad F_1^{(2n+\frac{3}{2})}, \quad F_3^{(2n+\frac{3}{2})}\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}M_1^{(2n+1)}, M_3^{(2n+1)}, \quad M_2^{(2n)}, M_4^{(2n)}, \quad K_1^{(2n)}, K_2^{(2n)}, I^{(2n)} \\ G_1^{(2n+\frac{3}{2})}, G_3^{(2n+\frac{3}{2})}, F_2^{(2n+\frac{3}{2})}, F_4^{(2n+\frac{3}{2})}, \\ G_2^{(2n+\frac{1}{2})}, G_4^{(2n+\frac{1}{2})}, \quad F_1^{(2n+\frac{1}{2})}, \quad F_3^{(2n+\frac{1}{2})}\end{aligned}$$

where $n \in \mathbb{Z}$.

Our reduction process is now facilitated by the fact that the set (3.10) of $\widehat{sl}(2|2)$ generators constitutes a subalgebra, which we denote by $\widehat{sl}_0(2|2)$. The $\widehat{sl}_0(2|2)$ subalgebra includes $E = K_1^{(1)} + K_2^{(1)} + I^{(1)}$ and the image of E within $\widehat{sl}_0(2|2)$ contains only the Cartan generators $M_1^{(2n)}$ and $M_3^{(2n)}$ among even grade elements. Notice that within the $\widehat{sl}_0(2|2)$ subalgebra the term k_0 from (2.13) vanishes identically.

We now associate the Lax operator

$$L = \partial_x + E + A_0 + A_{\frac{1}{2}} \tag{3.11}$$

to $\widehat{sl}_0(2|2)$ subalgebra by choosing

$$A_0 = u_1 M_1^{(0)} + u_3 M_3^{(0)}, \quad A_{\frac{1}{2}} = \bar{\psi}_1 G_1^{(\frac{1}{2})} + \bar{\psi}_3 G_3^{(\frac{1}{2})}$$

which are elements of $\widehat{sl}_0(2|2)$ with grade zero and $1/2$, respectively.

To find $N = 2$ mKdV equations we need to solve the zero-curvature equation (2.17) for $n = 3$. It is explicitly given by

$$\left[\partial_x + E + A_0 + A_{\frac{1}{2}}, \partial_3 + D_3^{(0)} + D_3^{(\frac{1}{2})} + D_3^{(1)} + D_3^{(\frac{3}{2})} + D_3^{(2)} + D_3^{(\frac{5}{2})} + E^{(3)} \right] = 0 \quad (3.12)$$

with $E^{(3)} = K_1^{(3)} + K_2^{(3)} + I_2^{(3)}$.

We found the following solution for the D 's within the $\widehat{sl}_0(2|2)$ subalgebra :

$$\begin{aligned} D_M^{(5/2)} &= \lambda^2 A_{\frac{1}{2}}, & D_K^{(5/2)} &= 0 \\ D_M^{(2)} &= \lambda^2 A_0, & D_K^{(2)} &= 0 \\ D_M^{(3/2)} &= \frac{1}{2} \left(\partial_x \psi_1 G_2^{(\frac{3}{2})} - \partial_x \psi_3 G_4^{(\frac{3}{2})} \right), & D_K^{(3/2)} &= d_1 F_1^{(\frac{3}{2})} + d_3 F_3^{(\frac{3}{2})} \\ D_M^{(1)} &= c_2 M_2^{(1)} + c_4 M_4^{(1)}, & D_K^{(1)} &= a_1 K_1^{(1)} + a_2 K_2^{(1)} + a_0 I_2^{(1)} \\ D_M^{(1/2)} &= \beta_1 G_1^{(\frac{1}{2})} + \beta_3 G_3^{(\frac{1}{2})}, & D_K^{(1/2)} &= \gamma_2 F_2^{(\frac{1}{2})} + \gamma_4 F_4^{(\frac{1}{2})} \\ D_M^{(0)} &= \delta_1 M_1^{(0)} + \delta_3 M_3^{(0)}, & D_K^{(0)} &= 0 \end{aligned}$$

where

$$\begin{aligned} d_1 &= -\frac{1}{2} (\psi_1 u_1 + \psi_3 u_3), & d_3 &= \frac{1}{2} (\psi_1 u_3 + \psi_3 u_1) \\ c_2 &= \frac{1}{2} u_1' - \psi_1 (\psi_3 u_3), & c_4 &= \frac{1}{2} u_3' + \psi_1 (\psi_3 u_1) \\ a_0 &= \frac{1}{2} (-\psi_1 \partial_x \psi_1 - \psi_3 \partial_x \psi_3), \\ a_1 &= \frac{1}{2} (\psi_1 \partial_x \psi_1 - \psi_3 \partial_x \psi_3 - u_1^2), & a_2 &= \frac{1}{2} (-\psi_1 \partial_x \psi_1 + \psi_3 \partial_x \psi_3 - u_3^2) \\ \beta_1 &= \frac{1}{4} \partial_x^2 \psi_1 - \frac{1}{2} \psi_1 (u_1^2 + u_3^2) - \frac{1}{2} \psi_3 (u_1 u_3), & \beta_3 &= \frac{1}{4} \partial_x^2 \psi_3 - \frac{1}{2} \psi_3 (u_1^2 + u_3^2) - \frac{1}{2} \psi_1 (u_1 u_3) \\ \gamma_2 &= \frac{1}{4} (\psi_1 u_1' - \psi_1' u_1 - \psi_3 u_3' + \psi_3' u_3), & \gamma_4 &= \frac{1}{4} (-\psi_1 u_3' + \psi_1' u_3 + \psi_3 u_1' - \psi_3' u_1) \end{aligned}$$

and

$$4\delta_1 = \partial_x^2 (u_1) - 2u_1^3 + 3u_1 (\psi_1 \partial_x \psi_1 - \psi_3 \partial_x \psi_3) + 3u_3 (-\psi_1 \partial_x \psi_3 + \psi_3 \partial_x \psi_1) \quad (3.13)$$

$$4\delta_3 = \partial_x^2 (u_3) - 2u_3^3 - 3u_3 (\psi_1 \partial_x \psi_1 - \psi_3 \partial_x \psi_3) - 3u_1 (-\psi_1 \partial_x \psi_3 + \psi_3 \partial_x \psi_1) \quad (3.14)$$

This leads to the following equations of motion :

$$4\partial_3 \psi_1 = \partial_x^3 \psi_1 - \frac{3}{2} \psi_1 \partial_x (u_1^2 + u_3^2) - 3\partial_x (\psi_1) (u_1^2 + u_3^2) - 3\psi_3 \partial_x (u_1 u_3) \quad (3.15)$$

$$4\partial_3 \psi_3 = \partial_x^3 \psi_3 - \frac{3}{2} \psi_3 \partial_x (u_1^2 + u_3^2) - 3\partial_x (\psi_3) (u_1^2 + u_3^2) - 3\psi_1 \partial_x (u_1 u_3) \quad (3.16)$$

for fermionic and :

$$\partial_3 u_i = \partial_x \delta_i, \quad i = 1, 3 \quad (3.17)$$

for bosonic modes. These are the $N = 2$ supersymmetric mKdV equations. They are invariant under the supersymmetry transformations

$$\partial_{\frac{1}{2}} u_1 = 2\partial_x (-\psi_1 \epsilon_2 + \psi_3 \epsilon_4), \quad \partial_{\frac{1}{2}} u_3 = 2\partial_x (-\psi_1 \epsilon_4 + \psi_3 \epsilon_2) \quad (3.18)$$

and

$$\partial_{\frac{1}{2}} \psi_1 = u_1 \epsilon_2 - u_3 \epsilon_4, \quad \partial_{\frac{1}{2}} \psi_3 = u_1 \epsilon_4 - u_3 \epsilon_2 \quad (3.19)$$

derived from the zero-curvature equation (2.16) with

$$D^{(\frac{1}{2})} = \epsilon_2 F_2^{(1/2)} + \epsilon_4 F_4^{(1/2)} \quad (3.20)$$

$$D^{(0)} = 2(-\psi_1 \epsilon_2 + \psi_3 \epsilon_4) M_1^{(0)} + 2(-\psi_1 \epsilon_4 + \psi_3 \epsilon_2) M_3^{(0)}. \quad (3.21)$$

Applying the transformations (3.18)-(3.19) twice we obtain:

$$\begin{aligned} \partial_{\frac{1}{2}}^2 u_1 &= -4\partial_x u_3 \epsilon_2 \epsilon_4, & \partial_{\frac{1}{2}}^2 u_3 &= -4\partial_x u_1 \epsilon_2 \epsilon_4, \\ \partial_{\frac{1}{2}}^2 \psi_1 &= -4\partial_x \psi_3 \epsilon_2 \epsilon_4, & \partial_{\frac{1}{2}}^2 \psi_3 &= -4\partial_x \psi_1 \epsilon_2 \epsilon_4. \end{aligned}$$

In terms of variables $u_{\pm} = u_1 \pm u_3$ and $\psi_{\pm} = \psi_1 \pm \psi_3$ this becomes

$$\partial_{\frac{1}{2}}^2 u_{\pm} = \mp 4\partial_x u_{\pm} \epsilon_2 \epsilon_4, \quad \partial_{\frac{1}{2}}^2 \psi_{\pm} = \mp 4\partial_x \psi_{\pm} \epsilon_2 \epsilon_4.$$

Each of elements in the kernel \mathcal{K} of E in $\widehat{sl}_0(2|2)$ gives rise to a symmetry transformation of the supersymmetric $N = 2$ mKdV hierarchy. We will now explore the structure of these symmetry transformations. Recall, that all symmetry transformations commute with isospectral deformations. The kernel elements among the algebra generators (3.10) give rise to the bosonic symmetry transformations $\delta_{K_i^{(2n+1)}}$ and $\delta_{K_1^{(2n+1)}}$ and the supersymmetry transformations of the form $\delta_{\epsilon F_1^{(2n+\frac{3}{2})}}$, $\delta_{\epsilon F_2^{(2n+\frac{1}{2})}}$, $\delta_{\epsilon F_3^{(2n+\frac{3}{2})}}$ and $\delta_{\epsilon F_4^{(2n+\frac{1}{2})}}$. Based on algebraic relations of generators we find that the commutation relations between the supersymmetry transformations and (in general non-local) bosonic symmetry transformations are:

$$\begin{aligned} \left[\delta_{K_i^{(2n+1)}}, \delta_{\epsilon F_1^{(2m+\frac{3}{2})}} \right] &= (-1)^i \delta_{\epsilon F_2^{(2(n+m+1)+\frac{1}{2})}}, & \left[\delta_{K_i^{(2n+1)}}, \delta_{\epsilon F_3^{(2m+\frac{3}{2})}} \right] &= (-1)^{i+1} \delta_{\epsilon F_4^{(2(n+m+1)+\frac{1}{2})}} \\ \left[\delta_{K_i^{(2n+1)}}, \delta_{\epsilon F_2^{(2m+\frac{1}{2})}} \right] &= (-1)^i \delta_{\epsilon F_1^{(2(n+m)+\frac{3}{2})}}, & \left[\delta_{K_i^{(2n+1)}}, \delta_{\epsilon F_4^{(2m+\frac{1}{2})}} \right] &= (-1)^{i+1} \delta_{\epsilon F_3^{(2(n+m)+\frac{3}{2})}} \end{aligned}$$

for $i = 1, 2$.

The commutation relations of the fermionic symmetry transformations takes a form of the following graded symmetry algebra :

$$\begin{aligned} \left[\delta_{\epsilon F_k^{(2n+\frac{3}{2})}}, \delta_{\bar{\epsilon} F_k^{(2m+\frac{3}{2})}} \right] &= 2\delta_{\epsilon \bar{\epsilon} E^{(2(n+m+1)+1)}}, \quad k = 1, 3 \\ \left[\delta_{\epsilon F_r^{(2n+\frac{1}{2})}}, \delta_{\bar{\epsilon} F_r^{(2m+\frac{1}{2})}} \right] &= -2\delta_{\epsilon \bar{\epsilon} E^{(2(n+m)+1)}}, \quad r = 2, 4 \end{aligned}$$

The above graded algebra represents the complete non-local symmetry structure of the supersymmetric $N = 2$ mKdV hierarchy.

We now turn our attention to the $N = 2$ Sine-Gordon model. The model is constructed in terms of the following objects :

$$A_0 = -\partial_x B B^{-1} = -\partial_x \phi_1 M_1^{(0)} - \partial_x \phi_3 M_3^{(0)} \quad (3.22)$$

$$B = e^{\phi_1 M_1^{(0)} + \phi_3 M_3^{(0)}} \quad (3.23)$$

$$A_{1/2} = \psi_1 G_1^{(1/2)} + \psi_3 G_3^{(1/2)} \quad (3.24)$$

$$J_{-1/2} = \psi_2 G_2^{(-1/2)} + \psi_4 G_4^{(-1/2)} \quad (3.25)$$

$$E^{(-1)} = K_1^{(-1)} + K_2^{(-1)} + I_2^{(-1)} \quad (3.26)$$

given in terms of generators from the $\widehat{sl}_0(2|2)$ subalgebra (3.10). In addition, we use $D^{(\frac{1}{2})}$ as given in (3.20).

We will use the following notation:

$$\bar{\psi}_{\pm} = \psi_2 \pm \psi_4, \quad \psi_{\pm} = \psi_1 \pm \psi_3, \quad \phi^{\pm} = \phi_1 \pm \phi_3.$$

Making use of

$$\begin{aligned} B j_{-1/2} B^{-1} &= e^{\phi_1 M_1 + \phi_3 M_3} \left(\psi_2 G_2^{(-1/2)} + \psi_4 G_4^{(-1/2)} \right) e^{-\phi_1 M_1 - \phi_3 M_3} \\ &= G_2^{(-1/2)} (\psi_2 \cosh \phi_3 \cosh \phi_1 - \psi_4 \sinh \phi_3 \sinh \phi_1) \\ &\quad + G_4^{(-1/2)} (-\psi_2 \sinh \phi_3 \sinh \phi_1 + \psi_4 \cosh \phi_3 \cosh \phi_1) \\ &\quad + F_1^{(-1/2)} (-\psi_2 \sinh \phi_1 \cosh \phi_3 + \psi_4 \cosh \phi_1 \sinh \phi_3) \\ &\quad + F_3^{(-1/2)} (\psi_2 \cosh \phi_1 \sinh \phi_3 - \psi_4 \sinh \phi_1 \cosh \phi_3) \end{aligned} \quad (3.27)$$

equations of motion (2.35) become

$$\partial_{-1} \psi_{\pm} = -2 \bar{\psi}_{\mp} \cosh \phi_{\pm}. \quad (3.28)$$

From equation (2.34) it follows that

$$\partial_x \bar{\psi}_{\pm} = -2 \psi_{\mp} \cosh \phi_{\pm} \quad (3.29)$$

In addition, we have

$$\partial_{-1} \partial_x \phi_+ = 4 \sinh \phi_+ \cosh \phi_- - 4 \psi_+ \bar{\psi}_+ \sinh \phi_- \quad (3.30)$$

$$\partial_{-1} \partial_x \phi_- = 4 \cosh \phi_+ \sinh \phi_- - 4 \psi_- \bar{\psi}_- \sinh \phi_+ \quad (3.31)$$

obtained from equation (2.36) using an identity :

$$\begin{aligned} B E^{(-1)} B^{-1} &= e^{\phi_1 M_1 + \phi_3 M_3} \left(K_1^{(-1)} + K_2^{(-1)} + K_3^{(-1)} \right) e^{-\phi_1 M_1 - \phi_3 M_3} \\ &= K_1^{(-1)} \cosh(2\phi_1) + K_2^{(-1)} \cosh(2\phi_3) \\ &\quad - M_2^{(-1)} \sinh(2\phi_1) - M_4^{(-1)} \sinh(2\phi_3) + K_3^{(-1)}. \end{aligned} \quad (3.32)$$

The above equations of motion are invariant under the supersymmetry transformations:

$$\delta\phi_{\pm} = 2\psi_{\mp}\epsilon_{\pm}, \quad \delta\psi_{\pm} = -\partial_x\phi_{\mp}\epsilon_{\pm} \quad (3.33)$$

where

$$\epsilon^{\pm} = \epsilon_2 \pm \epsilon_4.$$

Furthermore from equation (2.30) we get

$$\delta\bar{\psi}_{\pm} = 2\sinh\phi_{\pm}\epsilon_{\mp}, \quad (3.34)$$

The above $N = 2$ Sinh-Gordon equations coincide with ones proposed by Kobayashi and Uematsu in [19, 20] (see also the recent paper [21]) if we rescale E and $\psi, \bar{\psi}$.

4 N=4 mKdV equation from subalgebra of $\widehat{sl}(4|4)$

Based on the subalgebra of $\widehat{sl}(4, 4)$ generated by generators from formula (A.8) of Appendix A we define Lax operator as in (3.11) where

$$\begin{aligned} A_0 &= u_1 M_1^{(0)} + u_3 M_3^{(0)} + u_5 M_5^{(0)} + u_7 M_7^{(0)} \\ A_{\frac{1}{2}} &= \bar{\psi}_1 G_1^{(\frac{1}{2})} + \bar{\psi}_3 G_3^{(\frac{1}{2})} + \bar{\psi}_5 G_5^{(\frac{1}{2})} + \bar{\psi}_7 G_7^{(\frac{1}{2})} \end{aligned}$$

Notice that within the subalgebra generated by (A.8) the kernel component k_0 in the Lax operator vanishes identically. Next, we solve the zero-curvature equation (2.17) given for $n = 3$ explicitly by the commutation relation (3.12) with $E^{(3)} = K_1^{(3)} + K_2^{(3)} + I^{(3)}$.

We found the following solution for the D 's :

$$\begin{aligned} D^{(5/2)} &= D_M^{(5/2)} = \lambda^2 A_{\frac{1}{2}} \\ D_M^{(2)} &= \lambda^2 A_0, \quad D_K^{(2)} = 0 \\ D_M^{(3/2)} &= \frac{1}{2} \left(\partial_x \psi_1 G_2^{(\frac{3}{2})} - \partial_x \psi_3 G_4^{(\frac{3}{2})} - \partial_x \psi_5 G_6^{(\frac{3}{2})} + \partial_x \psi_7 G_8^{(\frac{3}{2})} \right) \\ D_K^{(3/2)} &= d_1 F_1^{(\frac{3}{2})} + d_3 F_3^{(\frac{3}{2})} + d_5 F_5^{(\frac{3}{2})} + d_7 F_7^{(\frac{3}{2})} \\ D_M^{(1)} &= c_2 M_2^{(1)} + c_4 M_4^{(1)} + c_6 M_6^{(1)} + c_8 M_8^{(1)} \\ D_K^{(1)} &= a_1 K_1^{(1)} + a_2 K_2^{(1)} + a_3 K_3^{(1)} + a_6 K_6^{(1)} + a_7 K_7^{(1)} + a_8 K_8^{(1)} + a_0 I^{(1)} \\ D_M^{(1/2)} &= \beta_1 G_1^{(\frac{1}{2})} + \beta_3 G_3^{(\frac{1}{2})} + \beta_5 G_5^{(\frac{1}{2})} + \beta_7 G_7^{(\frac{1}{2})} \\ D_K^{(1/2)} &= \gamma_2 F_2^{(\frac{1}{2})} + \gamma_4 F_4^{(\frac{1}{2})} + \gamma_6 F_6^{(\frac{1}{2})} + \gamma_8 F_8^{(\frac{1}{2})} \\ D_M^{(0)} &= \delta_1 M_1^{(0)} + \delta_3 M_3^{(0)} + \delta_5 M_5^{(0)} + \delta_7 M_7^{(0)} \\ D_K^{(0)} &= 0 \end{aligned} \quad (4.1)$$

where

$$\begin{aligned} d_1 &= -\frac{1}{2} (\psi_1 u_1 + \psi_3 u_3 + \psi_5 u_5 + \psi_7 u_7), & d_3 &= \frac{1}{2} (\psi_1 u_3 + \psi_3 u_1 + \psi_5 u_7 + \psi_7 u_5) \\ d_5 &= \frac{1}{2} (\psi_1 u_5 + \psi_3 u_7 + \psi_5 u_1 + \psi_7 u_3), & d_7 &= -\frac{1}{2} (\psi_1 u_7 + \psi_3 u_5 + \psi_5 u_3 + \psi_7 u_1) \end{aligned}$$

$$\begin{aligned}
c_2 &= \frac{1}{2}u'_1 - 2\psi_1(\psi_3u_3 + \psi_5u_5) + 2\psi_3\psi_7u_5 + 2\psi_5\psi_7u_3 \\
c_4 &= \frac{1}{2}u'_3 + 2\psi_1(\psi_3u_1 + \psi_5u_7) - 2\psi_3\psi_7u_7 - 2\psi_5\psi_7u_1 \\
c_6 &= \frac{1}{2}u'_5 + 2\psi_1(\psi_3u_7 + \psi_5u_1) - 2\psi_3\psi_7u_1 - 2\psi_5\psi_7u_7 \\
c_8 &= \frac{1}{2}u'_7 - 2\psi_1(\psi_3u_5 + \psi_5u_3) + 2\psi_3\psi_7u_3 + 2\psi_5\psi_7u_5
\end{aligned}$$

$$\begin{aligned}
a_0 &= -\psi_1\partial_x\psi_1 - \psi_3\partial_x\psi_3 - \psi_5\partial_x\psi_5 - \psi_7\partial_x\psi_7 \\
a_1 &= \psi_1\partial_x\psi_1 - \psi_3\partial_x\psi_3 - \psi_5\partial_x\psi_5 + \psi_7\partial_x\psi_7 - \frac{1}{2}u_1^2 - \frac{1}{2}u_7^2 \\
a_2 &= -\psi_1\partial_x\psi_1 + \psi_3\partial_x\psi_3 + \psi_5\partial_x\psi_5 - \psi_7\partial_x\psi_7 - \frac{1}{2}u_3^2 - \frac{1}{2}u_5^2 \\
a_3 &= \alpha_7 = -\psi_1\partial_x\psi_7 - \psi_3\partial_x\psi_5 - \psi_5\partial_x\psi_3 - \psi_7\partial_x\psi_1 \\
a_6 &= -\psi_1\partial_x\psi_7 + \psi_3\partial_x\psi_5 + \psi_5\partial_x\psi_3 - \psi_7\partial_x\psi_1 - u_3u_5 \\
a_8 &= \psi_1\partial_x\psi_7 - \psi_3\partial_x\psi_5 - \psi_5\partial_x\psi_3 + \psi_7\partial_x\psi_1 - u_1u_7
\end{aligned}$$

$$\begin{aligned}
\beta_1 &= \frac{1}{4}\partial_x^2\psi_1 - \frac{1}{2}\psi_1(u_1^2 + u_3^2 + u_5^2 + u_7^2) - \frac{1}{2}\psi_3(u_1u_3 + u_5u_7) \\
&\quad - \frac{1}{2}\psi_5(u_1u_5 + u_3u_7) - \psi_7(u_3u_5 + u_1u_7) \\
\beta_3 &= \frac{1}{4}\partial_x^2\psi_3 - \frac{1}{2}\psi_3(u_1^2 + u_3^2 + u_5^2 + u_7^2) - \frac{1}{2}\psi_1(u_1u_3 + u_5u_7) \\
&\quad - \frac{1}{2}\psi_7(u_1u_5 + u_3u_7) - \psi_5(u_3u_5 + u_1u_7) \\
\beta_5 &= \frac{1}{4}\partial_x^2\psi_5 - \frac{1}{2}\psi_5(u_1^2 + u_3^2 + u_5^2 + u_7^2) - \frac{1}{2}\psi_7(u_1u_3 + u_5u_7) \\
&\quad - \frac{1}{2}\psi_1(u_1u_5 + u_3u_7) - \psi_3(u_3u_5 + u_1u_7) \\
\beta_7 &= \frac{1}{4}\partial_x^2\psi_7 - \frac{1}{2}\psi_7(u_1^2 + u_3^2 + u_5^2 + u_7^2) - \frac{1}{2}\psi_5(u_1u_3 + u_5u_7) \\
&\quad - \frac{1}{2}\psi_3(u_1u_5 + u_3u_7) - \psi_1(u_3u_5 + u_1u_7)
\end{aligned}$$

$$\begin{aligned}
\gamma_2 &= \frac{1}{4}(\psi_1u'_1 - \psi'_1u_1 - \psi_3u'_3 + \psi'_3u_3 - \psi_5u'_5 + \psi'_5u_5 + \psi_7u'_7 - \psi'_7u_7) \\
\gamma_4 &= \frac{1}{4}(-\psi_1u'_3 + \psi'_1u_3 + \psi_3u'_1 - \psi'_3u_1 + \psi_5u'_7 - \psi'_5u_7 - \psi_7u'_5 + \psi'_7u_5) \\
\gamma_6 &= \frac{1}{4}(-\psi_1u'_5 + \psi'_1u_5 + \psi_3u'_7 - \psi'_3u_7 + \psi_5u'_1 - \psi'_5u_1 - \psi_7u'_3 + \psi'_7u_3) \\
\gamma_8 &= \frac{1}{4}(\psi_1u'_7 - \psi'_1u_7 - \psi_3u'_5 + \psi'_3u_5 - \psi_5u'_3 + \psi'_5u_3 + \psi_7u'_1 - \psi'_7u_1)
\end{aligned}$$

$$\begin{aligned}
4\delta_1 &= \partial_x^2(u_1) - 2u_1(u_1^2 + 3u_7^2) + 3u_1(\psi_1\partial_x\psi_1 - \psi_3\partial_x\psi_3 - \psi_5\partial_x\psi_5 + \psi_7\partial_x\psi_7) \\
&+ 3u_3(-\psi_1\partial_x\psi_3 + \psi_3\partial_x\psi_1 + \psi_5\partial_x\psi_7 - \psi_7\partial_x\psi_5) \\
&+ 3u_5(-\psi_1\partial_x\psi_5 + \psi_3\partial_x\psi_7 + \psi_5\partial_x\psi_1 - \psi_7\partial_x\psi_3) \\
&+ 3u_7(\psi_1\partial_x\psi_7 - \psi_3\partial_x\psi_5 - \psi_5\partial_x\psi_3 + \psi_7\partial_x\psi_1)
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
4\delta_3 &= \partial_x^2(u_3) - 2u_3(u_3^2 + 3u_5^2) - 3u_3(\psi_1\partial_x\psi_1 - \psi_3\partial_x\psi_3 - \psi_5\partial_x\psi_5 + \psi_7\partial_x\psi_7) \\
&- 3u_1(-\psi_1\partial_x\psi_3 + \psi_3\partial_x\psi_1 + \psi_5\partial_x\psi_7 - \psi_7\partial_x\psi_5) \\
&- 3u_7(-\psi_1\partial_x\psi_5 + \psi_3\partial_x\psi_7 + \psi_5\partial_x\psi_1 - \psi_7\partial_x\psi_3) \\
&- 3u_5(\psi_1\partial_x\psi_7 - \psi_3\partial_x\psi_5 - \psi_5\partial_x\psi_3 + \psi_7\partial_x\psi_1)
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
4\delta_5 &= \partial_x^2(u_5) - 2u_5(u_5^2 + 3u_3^2) - 3u_5(\psi_1\partial_x\psi_1 - \psi_3\partial_x\psi_3 - \psi_5\partial_x\psi_5 + \psi_7\partial_x\psi_7) \\
&- 3u_7(-\psi_1\partial_x\psi_3 + \psi_3\partial_x\psi_1 + \psi_5\partial_x\psi_7 - \psi_7\partial_x\psi_5) \\
&- 3u_1(-\psi_1\partial_x\psi_5 + \psi_3\partial_x\psi_7 + \psi_5\partial_x\psi_1 - \psi_7\partial_x\psi_3) \\
&- 3u_3(\psi_1\partial_x\psi_7 - \psi_3\partial_x\psi_5 - \psi_5\partial_x\psi_3 + \psi_7\partial_x\psi_1)
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
4\delta_7 &= \partial_x^2(u_7) - 2u_7(u_7^2 + 3u_1^2) + 3u_7(\psi_1\partial_x\psi_1 - \psi_3\partial_x\psi_3 - \psi_5\partial_x\psi_5 + \psi_7\partial_x\psi_7) \\
&+ 3u_5(-\psi_1\partial_x\psi_3 + \psi_3\partial_x\psi_1 + \psi_5\partial_x\psi_7 - \psi_7\partial_x\psi_5) \\
&+ 3u_3(-\psi_1\partial_x\psi_5 + \psi_3\partial_x\psi_7 + \psi_5\partial_x\psi_1 - \psi_7\partial_x\psi_3) \\
&+ 3u_1(\psi_1\partial_x\psi_7 - \psi_3\partial_x\psi_5 - \psi_5\partial_x\psi_3 + \psi_7\partial_x\psi_1)
\end{aligned} \tag{4.5}$$

This leads to the following equations of motion for fermions :

$$\begin{aligned}
4\partial_3\psi_1 &= \partial_x^3\psi_1 - \frac{3}{2}\psi_1\partial_x(u_1^2 + u_3^2 + u_5^2 + u_7^2) - 3\partial_x(\psi_1)(u_1^2 + u_3^2 + u_5^2 + u_7^2) \\
&- 3\psi_3\partial_x(u_1u_3 + u_5u_7) - 3\psi_5\partial_x(u_1u_5 + u_3u_7) \\
&- 3\psi_7\partial_x(u_3u_5 + u_1u_7) - 6\partial_x(\psi_7)(u_3u_5 + u_1u_7)
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
4\partial_3\psi_3 &= \partial_x^3\psi_3 - \frac{3}{2}\psi_3\partial_x(u_1^2 + u_3^2 + u_5^2 + u_7^2) - 3\partial_x(\psi_3)(u_1^2 + u_3^2 + u_5^2 + u_7^2) \\
&- 3\psi_1\partial_x(u_1u_3 + u_5u_7) - 3\psi_7\partial_x(u_1u_5 + u_3u_7) \\
&- 3\psi_5\partial_x(u_3u_5 + u_1u_7) - 6\partial_x(\psi_5)(u_3u_5 + u_1u_7)
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
4\partial_3\psi_5 &= \partial_x^3\psi_5 - \frac{3}{2}\psi_5\partial_x(u_1^2 + u_3^2 + u_5^2 + u_7^2) - 3\partial_x(\psi_5)(u_1^2 + u_3^2 + u_5^2 + u_7^2) \\
&- 3\psi_7\partial_x(u_1u_3 + u_5u_7) - 3\psi_1\partial_x(u_1u_5 + u_3u_7) \\
&- 3\psi_3\partial_x(u_3u_5 + u_1u_7) - 6\partial_x(\psi_3)(u_3u_5 + u_1u_7)
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
4\partial_3\psi_7 &= \partial_x^3\psi_7 - \frac{3}{2}\psi_7\partial_x(u_1^2 + u_3^2 + u_5^2 + u_7^2) - 3\partial_x(\psi_7)(u_1^2 + u_3^2 + u_5^2 + u_7^2) \\
&- 3\psi_5\partial_x(u_1u_3 + u_5u_7) - 3\psi_3\partial_x(u_1u_5 + u_3u_7) \\
&- 3\psi_1\partial_x(u_3u_5 + u_1u_7) - 6\partial_x(\psi_1)(u_3u_5 + u_1u_7)
\end{aligned} \tag{4.9}$$

and bosonic equations of motion:

$$\partial_3u_i = \partial_x\delta_i, \quad i = 1, 3, 5, 7 \tag{4.10}$$

These are the $N = 4$ supersymmetric mKdV equations.

They are invariant under the supersymmetry transformations derived from the zero-curvature equation (2.16) with

$$D^{(\frac{1}{2})} = \epsilon_2 F_2^{(1/2)} + \epsilon_4 F_4^{(1/2)} + \epsilon_6 F_6^{(1/2)} + \epsilon_8 F_8^{(1/2)} \quad (4.11)$$

$$D^{(0)} = \bar{a}_1 M_1^{(0)} + \bar{a}_3 M_3^{(0)} + \bar{a}_5 M_5^{(0)} + \bar{a}_7 M_7^{(0)} \quad (4.12)$$

with

$$\bar{a}_1 = 2(-\psi_1 \epsilon_2 + \psi_3 \epsilon_4 + \psi_5 \epsilon_6 - \psi_7 \epsilon_8)$$

$$\bar{a}_3 = 2(-\psi_1 \epsilon_4 + \psi_3 \epsilon_2 + \psi_5 \epsilon_8 - \psi_7 \epsilon_6)$$

$$\bar{a}_5 = 2(-\psi_1 \epsilon_6 + \psi_3 \epsilon_8 + \psi_5 \epsilon_2 - \psi_7 \epsilon_4)$$

$$\bar{a}_7 = 2(-\psi_1 \epsilon_8 + \psi_3 \epsilon_6 + \psi_5 \epsilon_4 - \psi_7 \epsilon_2)$$

The supersymmetry transformations are

$$\partial_{\frac{1}{2}} u_i = \partial_x \bar{a}_i, \quad i = 1, 3, 5, 7$$

and

$$\partial_{\frac{1}{2}} \psi_1 = u_1 \epsilon_2 - u_3 \epsilon_4 - u_5 \epsilon_6 + u_7 \epsilon_8$$

$$\partial_{\frac{1}{2}} \psi_3 = u_1 \epsilon_4 - u_3 \epsilon_2 - u_5 \epsilon_8 + u_7 \epsilon_6$$

$$\partial_{\frac{1}{2}} \psi_5 = u_1 \epsilon_6 - u_3 \epsilon_8 - u_5 \epsilon_2 + u_7 \epsilon_4$$

$$\partial_{\frac{1}{2}} \psi_7 = u_1 \epsilon_8 - u_3 \epsilon_6 - u_5 \epsilon_4 + u_7 \epsilon_2$$

5 N=4 Sinh-Gordon Model from subalgebra of $\widehat{sl}(4|4)$

Using the same subalgebra (A.8) of $\widehat{sl}(4|4)$ we find

$$A_0 = \bar{\partial} B B^{-1} = \bar{\partial} \phi_1 M_1 + \bar{\partial} \phi_3 M_3 + \bar{\partial} \phi_5 M_5 + \bar{\partial} \phi_7 M_7 \quad (5.1)$$

$$B = e^{\phi_1 M_1 + \phi_3 M_3 + \phi_5 M_5 + \phi_7 M_7} \quad (5.2)$$

$$A_{1/2} = \psi_1 G_1^{(1/2)} + \psi_3 G_3^{(1/2)} + \psi_5 G_5^{(1/2)} + \psi_7 G_7^{(1/2)} \quad (5.3)$$

$$J_{-1/2} = \psi_2 G_2^{(-1/2)} + \psi_4 G_4^{(-1/2)} + \psi_6 G_6^{(-1/2)} + \psi_8 G_8^{(-1/2)} \quad (5.4)$$

with $D^{(\frac{1}{2})}$ as in (4.11) and $E^{(-1)}$ as in (3.26).

In notation:

$$\bar{\psi}_{\pm} = \psi_2 \pm \psi_4, \quad \psi_{\pm} = \psi_1 \pm \psi_3, \quad \chi_{\pm} = \psi_5 \pm \psi_7, \quad \bar{\chi}_{\pm} = \psi_6 \pm \psi_8$$

$$\phi^{\pm} = \phi_1 \pm \phi_3, \quad \bar{\phi}^{\pm} = \phi_5 \pm \phi_7$$

equations of motion obtained from equation (2.35) become :

$$\partial_{-1} \psi_{\pm} = -2\bar{\psi}_{\mp} \cosh \phi_{\pm} \cosh \bar{\phi}_{\pm} + 2\bar{\chi}_{\mp} \sinh \phi_{\pm} \sinh \bar{\phi}_{\pm} \quad (5.5)$$

$$\partial_{-1} \chi_{\pm} = -2\bar{\psi}_{\mp} \sinh \phi_{\pm} \sinh \bar{\phi}_{\pm} + 2\bar{\chi}_{\mp} \cosh \phi_{\pm} \cosh \bar{\phi}_{\pm} \quad (5.6)$$

From equation (2.34) it follows that

$$\partial_x \bar{\psi}_\pm = -2\psi_\mp \cosh \phi_\pm \cosh \bar{\phi}_\pm + 2\chi_\mp \sinh \phi_\pm \sinh \bar{\phi}_\pm \quad (5.7)$$

$$\partial_x \bar{\chi}_\pm = 2\chi_\mp \cosh \phi_\pm \cosh \bar{\phi}_\pm - 2\psi_\mp \sinh \phi_\pm \sinh \bar{\phi}_\pm \quad (5.8)$$

From (2.36) we obtain :

$$\begin{aligned} \partial_{-1} \partial_x \phi_+ &= 4 \left[\sinh \phi_+ \cosh \phi_- \cosh \bar{\phi}_+ \cosh \bar{\phi}_- - \cosh \phi_+ \sinh \phi_- \sinh \bar{\phi}_+ \sinh \bar{\phi}_- \right] \\ &+ 4\psi_+ \left(-\bar{\psi}_+ \sinh \phi_- \cosh \bar{\phi}_- + \bar{\chi}_+ \cosh \phi_- \sinh \bar{\phi}_- \right) \\ &+ 4\chi_+ \left(\bar{\psi}_+ \cosh \phi_- \sinh \bar{\phi}_- - \bar{\chi}_+ \sinh \phi_- \cosh \bar{\phi}_- \right) \end{aligned} \quad (5.9)$$

$$\begin{aligned} \partial_{-1} \partial_x \phi_- &= -4 \left[\sinh \phi_+ \cosh \phi_- \sinh \bar{\phi}_+ \sinh \bar{\phi}_- - \cosh \phi_+ \sinh \phi_- \cosh \bar{\phi}_+ \cosh \bar{\phi}_- \right] \\ &+ 4\psi_- \left(-\bar{\psi}_- \sinh \phi_+ \cosh \bar{\phi}_+ + \bar{\chi}_- \cosh \phi_+ \sinh \bar{\phi}_+ \right) \\ &+ 4\chi_- \left(\bar{\psi}_- \cosh \phi_+ \sinh \bar{\phi}_+ - \bar{\chi}_- \sinh \phi_+ \cosh \bar{\phi}_+ \right) \end{aligned} \quad (5.10)$$

and

$$\begin{aligned} \partial_{-1} \partial_x \bar{\phi}_+ &= 4 \left[\cosh \phi_+ \cosh \phi_- \sinh \bar{\phi}_+ \cosh \bar{\phi}_- - \sinh \phi_+ \sinh \phi_- \cosh \bar{\phi}_+ \sinh \bar{\phi}_- \right] \\ &+ 4\psi_+ \left(\bar{\psi}_+ \cosh \phi_- \sinh \bar{\phi}_- - \bar{\chi}_+ \sinh \phi_- \cosh \bar{\phi}_- \right) \\ &+ 4\chi_+ \left(-\bar{\psi}_+ \sinh \phi_- \cosh \bar{\phi}_- + \bar{\chi}_+ \cosh \phi_- \sinh \bar{\phi}_- \right) \end{aligned} \quad (5.11)$$

$$\begin{aligned} \partial_{-1} \partial_x \bar{\phi}_- &= 4 \left[\cosh \phi_+ \cosh \phi_- \cosh \bar{\phi}_+ \sinh \bar{\phi}_- - \sinh \phi_+ \sinh \phi_- \sinh \bar{\phi}_+ \cosh \bar{\phi}_- \right] \\ &+ 4\psi_- \left(\bar{\psi}_- \cosh \phi_+ \sinh \bar{\phi}_+ - \bar{\chi}_- \sinh \phi_+ \cosh \bar{\phi}_+ \right) \\ &+ 4\chi_- \left(-\bar{\psi}_- \sinh \phi_+ \cosh \bar{\phi}_+ + \bar{\chi}_- \cosh \phi_+ \sinh \bar{\phi}_+ \right) \end{aligned} \quad (5.12)$$

These equations are invariant under supersymmetry transformations:

$$\delta \phi_\pm = 2\psi_\mp \epsilon_\pm - 2\chi_\mp \bar{\epsilon}_\pm, \quad \delta \bar{\phi}_\pm = 2\psi_\mp \bar{\epsilon}_\pm - 2\chi_\mp \epsilon_\pm,$$

and

$$\delta \psi_\pm = -\partial_x \phi_\mp \epsilon_\pm + \partial_x \bar{\phi}_\mp \bar{\epsilon}_\pm, \quad \delta \chi_\pm = \partial_x \bar{\phi}_\mp \epsilon_\pm - \partial_x \phi_\mp \bar{\epsilon}_\pm$$

where

$$\epsilon^\pm = \epsilon_2 \pm \epsilon_4, \quad \bar{\epsilon}_\pm = \epsilon_6 \pm \epsilon_8$$

Furthermore from (2.30) we get

$$\delta \bar{\psi}_\pm = 2 \sinh \phi_\pm \cosh \bar{\phi}_\pm \epsilon_\mp - 2 \cosh \phi_\pm \sinh \bar{\phi}_\pm \bar{\epsilon}_\mp,$$

$$\delta \bar{\chi}_\pm = 2 \cosh \phi_\pm \sinh \bar{\phi}_\pm \epsilon_\mp - 2 \sinh \phi_\pm \cosh \bar{\phi}_\pm \bar{\epsilon}_\mp$$

In the limit

$$\bar{\phi}_\pm = \chi_\pm = \bar{\chi}_\pm = 0$$

the above equations reproduce the equations of motion (3.28), (3.29), (3.30) and (3.31) of $N = 2$ model and corresponding supersymmetry transformations (3.33) and (3.34).

6 Concluding Remarks

We offer here some summarizing comments on results obtained in the previous sections. Deriving extended supersymmetry structure without non-local terms was accomplished here due to two technical but crucial choices concerning the underlying algebraic structure. One choice was to base our construction on the loop algebras $\widehat{sl}(n|m)$ with $n = m$ leading to the identity I becoming a part of algebra. The second choice concerned reduction process. It was made in such a way as to ensure the Lax operator only contained Cartan generators among the zero-grade terms. We now briefly point out consequences of these two steps.

First, notice that the structure of the supersymmetry transformation is directly related to $\mathcal{K}_{\frac{1}{2}}$; the $\frac{1}{2}$ grade sector of the kernel \mathcal{K} . For the $N = 2$ case the relevant superalgebra is $sl(2, 2)$ which has 2 generators spanning $\mathcal{K}_{\frac{1}{2}}$, namely $F_2^{(\frac{1}{2})}$ and $F_4^{(\frac{1}{2})}$ both squaring to E modulo the identity element I , i.e.

$$(F_2^{(\frac{1}{2})})^2 = -E, \quad (F_4^{(\frac{1}{2})})^2 = E - 2I \quad (6.1)$$

For the $N = 4$ we have determined that the relevant subalgebra is composed of a semi direct product of $U(1)$ with $sl(2, 2) \otimes sl(2, 2)$. Each $sl(2, 2)$ subalgebra generates a subsector of grade $\frac{1}{2}$ of the kernel $\mathcal{K}_{\frac{1}{2}}$ like (6.1) such that $\mathcal{K}_{\frac{1}{2}}$ is spanned by $F_2^{(\frac{1}{2})}, F_4^{(\frac{1}{2})}, F_6^{(\frac{1}{2})}$ and $F_8^{(\frac{1}{2})}$ with

$$(F_2^{(\frac{1}{2})})^2 = (F_8^{(\frac{1}{2})})^2 = -E, \quad (F_4^{(\frac{1}{2})})^2 = (F_6^{(\frac{1}{2})})^2 = E - 2I \quad (6.2)$$

Since, $\delta_I = 0$ in the sense of the definition (2.1) we see that all the algebra elements in equations (6.1)-(6.2) give rise to identical (up to the sign) symmetry transformations.

Moreover, we observe that the same pattern emerges for higher supersymmetries (e.g. $N = 2n = 8$) where there is a natural decomposition of super Lie algebra $sl(2n, 2n)$ into semi direct product of $U(1)$ with $\otimes^n sl(2, 2)$.

Finally, going back to the reduction process we observe that within the relevant subalgebra structure the k_0 term in the Lax (2.14) vanishes identically. This is verified explicitly for the $sl(2, 2)$ case in equation (3.11) and also for the $sl(4, 4)$ case. This is an important feature, since the non-zero k_0 leads to the undesired non-local supersymmetry transformations. This feature, in fact, provided us with a guideline for choosing the relevant subalgebra.

Presently, we are studying the conservation laws for these models according to refs. [18] and [1]. Moreover, the generalization to higher N supersymmetric models, in particular for $N = 8$, is under investigation.

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A Appendix - The algebra of $\widehat{sl}(4, 4)$

A.1 $\widehat{sl}(4|4)$

The roots of $sl(4|4)$ are given by

$$\begin{aligned}\alpha_1 &= e_1 - e_2, & \alpha_2 &= e_2 - e_3, & \alpha_3 &= e_3 - e_4, & \alpha_4 &= e_4 - f_1, \\ \alpha_5 &= f_1 - f_2, & \alpha_6 &= f_2 - f_3, & \alpha_7 &= f_3 - f_4,\end{aligned}\tag{A.1}$$

where $e_i \cdot e_j = \delta_{ij}$, $f_i \cdot f_j = -\delta_{ij}$.

The grade of step operators $E_\beta^{(n)}$ of corresponding loop algebra is given by

$$\text{Grade}(E_\beta^{(n)}) = n + \frac{1}{2}(e_1 - e_2 + e_3 - e_4 + f_1 - f_2 + f_3 - f_4) \cdot \beta\tag{A.2}$$

Define Kernel of E to be:

$$\begin{aligned}f_{1,\eta}^{(n+\frac{1}{2})} &= (\eta E_{\alpha_4}^{(n+\frac{1}{2})} + E_{-\alpha_4}^{(n+\frac{1}{2})}) + (\eta E_{\alpha_3+\alpha_4+\alpha_5}^{(n+\frac{1}{2})} + E_{-\alpha_3-\alpha_4-\alpha_5}^{(n+\frac{1}{2})}), \\ f_{2,\eta}^{(n+\frac{1}{2})} &= (\eta E_{\alpha_3+\alpha_4}^{(n-\frac{1}{2})} + E_{-\alpha_3-\alpha_4}^{(n+\frac{3}{2})}) + (\eta E_{\alpha_4+\alpha_5}^{(n+\frac{3}{2})} + E_{-\alpha_4-\alpha_5}^{(n-\frac{1}{2})}), \\ f_{3,\eta}^{(n+\frac{1}{2})} &= (\eta E_{\alpha_2+\alpha_3+\alpha_4}^{(n+\frac{1}{2})} + E_{-\alpha_2-\alpha_3-\alpha_4}^{(n+\frac{1}{2})}) + (\eta E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5}^{(n+\frac{1}{2})} + E_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5}^{(n+\frac{1}{2})}), \\ f_{4,\eta}^{(n+\frac{1}{2})} &= (\eta E_{\alpha_4+\alpha_5+\alpha_6}^{(n+\frac{1}{2})} + E_{-\alpha_4-\alpha_5-\alpha_6}^{(n+\frac{1}{2})}) + (\eta E_{\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}^{(n+\frac{1}{2})} + E_{-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7}^{(n+\frac{1}{2})}), \\ f_{5,\eta}^{(n+\frac{1}{2})} &= (\eta E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}^{(n-\frac{1}{2})} + E_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}^{(n+\frac{3}{2})}) + (\eta E_{\alpha_2+\alpha_3+\alpha_4+\alpha_5}^{(n+\frac{3}{2})} + E_{-\alpha_2-\alpha_3-\alpha_4-\alpha_5}^{(n-\frac{1}{2})}), \\ f_{6,\eta}^{(n+\frac{1}{2})} &= (\eta E_{\alpha_4+\alpha_5+\alpha_6+\alpha_7}^{(n-\frac{1}{2})} + E_{-\alpha_4-\alpha_5-\alpha_6-\alpha_7}^{(n+\frac{3}{2})}) + (\eta E_{\alpha_3+\alpha_4+\alpha_5+\alpha_6}^{(n+\frac{3}{2})} + E_{-\alpha_3-\alpha_4-\alpha_5-\alpha_6}^{(n-\frac{1}{2})}), \\ f_{7,\eta}^{(n+\frac{1}{2})} &= (\eta E_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6}^{(n+\frac{1}{2})} + E_{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6}^{(n+\frac{1}{2})}) \\ &\quad + (\eta E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}^{(n+\frac{1}{2})} + E_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7}^{(n+\frac{1}{2})}), \\ f_{8,\eta}^{(n+\frac{1}{2})} &= (\eta E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6}^{(n-\frac{1}{2})} + E_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6}^{(n+\frac{3}{2})}) \\ &\quad + (\eta E_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}^{(n+\frac{3}{2})} + E_{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7}^{(n-\frac{1}{2})}),\end{aligned}$$

for $\eta = \pm 1$.

$$\begin{aligned}
K_1^{(n)} &= \left(E_{\alpha_7}^{(n+1)} + E_{-\alpha_7}^{(n-1)} \right) + \left(E_{\alpha_5}^{(n+1)} + E_{-\alpha_5}^{(n-1)} \right), \\
K_2^{(n)} &= \left(E_{\alpha_1}^{(n-1)} + E_{-\alpha_1}^{(n+1)} \right) + \left(E_{\alpha_3}^{(n-1)} + E_{-\alpha_3}^{(n+1)} \right), \\
K_3^{(n)} &= -(\alpha_5 + 2\alpha_6 + \alpha_7) \cdot H^{(n)}, \\
K_6^{(n)} &= \left(E_{\alpha_1}^{(n-1)} + E_{-\alpha_1}^{(n+1)} \right) - \left(E_{\alpha_3}^{(n-1)} + E_{-\alpha_3}^{(n+1)} \right), \\
K_7^{(n)} &= (\alpha_1 + 2\alpha_2 + \alpha_3) \cdot H^{(n)}, \\
K_8^{(n)} &= -\left(E_{\alpha_7}^{(n+1)} + E_{-\alpha_7}^{(n-1)} \right) + \left(E_{\alpha_5}^{(n+1)} + E_{-\alpha_5}^{(n-1)} \right), \\
I^{(n)} &= (\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7) \cdot H^{(n)}, \\
K_9^{\pm(n)} &= (E_{\alpha_1+\alpha_2}^{(n)} \pm E_{-\alpha_1-\alpha_2}^{(n)}) + (E_{\alpha_2+\alpha_3}^{(n)} \pm E_{-\alpha_2-\alpha_3}^{(n)}), \\
(K_{10}^{\pm})^{(n)} &= (E_{\alpha_5+\alpha_6}^{(n)} \pm E_{-\alpha_5-\alpha_6}^{(n)}) + (E_{\alpha_6+\alpha_7}^{(n)} \pm E_{-\alpha_6-\alpha_7}^{(n)}), \\
(K_{11}^{\pm})^{(n)} &= (E_{\alpha_1+\alpha_2+\alpha_3}^{(n-1)} \pm E_{-\alpha_1-\alpha_2-\alpha_3}^{(n+1)}) + (E_{\alpha_2}^{(n+1)} \pm E_{-\alpha_2}^{(n-1)}), \\
(K_{12}^{\pm})^{(n)} &= (E_{\alpha_5+\alpha_6+\alpha_7}^{(n+1)} \pm E_{-\alpha_5-\alpha_6-\alpha_7}^{(n-1)}) + (E_{\alpha_5}^{(n+1)} \pm E_{-\alpha_5}^{(n-1)}), \tag{A.3}
\end{aligned}$$

and Image of E

$$\begin{aligned}
g_{1,\eta}^{(n+\frac{1}{2})} &= (\eta E_{\alpha_4}^{(n+\frac{1}{2})} + E_{-\alpha_4}^{(n+\frac{1}{2})}) - (\eta E_{\alpha_3+\alpha_4+\alpha_5}^{(n+\frac{1}{2})} + E_{-\alpha_3-\alpha_4-\alpha_5}^{(n+\frac{1}{2})}), \\
g_{2,\eta}^{(n+\frac{1}{2})} &= (\eta E_{\alpha_3+\alpha_4}^{(n-\frac{1}{2})} + E_{-\alpha_3-\alpha_4}^{(n+\frac{3}{2})}) - (\eta E_{\alpha_4+\alpha_5}^{(n+\frac{3}{2})} + E_{-\alpha_4-\alpha_5}^{(n-\frac{1}{2})}), \\
g_{3,\eta}^{(n+\frac{1}{2})} &= (\eta E_{\alpha_2+\alpha_3+\alpha_4}^{(n+\frac{1}{2})} + E_{-\alpha_2-\alpha_3-\alpha_4}^{(n+\frac{1}{2})}) - (\eta E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5}^{(n+\frac{1}{2})} + E_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5}^{(n+\frac{1}{2})}), \\
g_{4,\eta}^{(n+\frac{1}{2})} &= (\eta E_{\alpha_4+\alpha_5+\alpha_6}^{(n+\frac{1}{2})} + E_{-\alpha_4-\alpha_5-\alpha_6}^{(n+\frac{1}{2})}) - (\eta E_{\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}^{(n+\frac{1}{2})} + E_{-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7}^{(n+\frac{1}{2})}), \\
g_{5,\eta}^{(n+\frac{1}{2})} &= (\eta E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}^{(n-\frac{1}{2})} + E_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4}^{(n+\frac{3}{2})}) - (\eta E_{\alpha_2+\alpha_3+\alpha_4+\alpha_5}^{(n+\frac{3}{2})} + E_{-\alpha_2-\alpha_3-\alpha_4-\alpha_5}^{(n-\frac{1}{2})}), \\
g_{6,\eta}^{(n+\frac{1}{2})} &= (\eta E_{\alpha_4+\alpha_5+\alpha_6+\alpha_7}^{(n-\frac{1}{2})} + E_{-\alpha_4-\alpha_5-\alpha_6-\alpha_7}^{(n+\frac{3}{2})}) - (\eta E_{\alpha_3+\alpha_4+\alpha_5+\alpha_6}^{(n+\frac{3}{2})} + E_{-\alpha_3-\alpha_4-\alpha_5-\alpha_6}^{(n-\frac{1}{2})}), \\
g_{7,\eta}^{(n+\frac{1}{2})} &= (\eta E_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6}^{(n+\frac{1}{2})} + E_{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6}^{(n+\frac{1}{2})}) \\
&\quad - (\eta E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}^{(n+\frac{1}{2})} + E_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7}^{(n+\frac{1}{2})}), \\
g_{8,\eta}^{(n+\frac{1}{2})} &= (\eta E_{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6}^{(n-\frac{1}{2})} + E_{-\alpha_1-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6}^{(n+\frac{3}{2})}) \\
&\quad - (\eta E_{\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7}^{(n+\frac{3}{2})} + E_{-\alpha_2-\alpha_3-\alpha_4-\alpha_5-\alpha_6-\alpha_7}^{(n-\frac{1}{2})}),
\end{aligned}$$

$$\begin{aligned}
M_1^{(n)} &= -(\alpha_5 + \alpha_7) \cdot H^{(n)}, \\
M_2^{(n)} &= \left(E_{\alpha_7}^{(n+1)} - E_{-\alpha_7}^{(n-1)}\right) + \left(E_{\alpha_5}^{(n+1)} - E_{-\alpha_5}^{(n-1)}\right), \\
M_3^{(n)} &= -(\alpha_1 + \alpha_3) \cdot H^{(n)}, \\
M_4^{(n)} &= -\left(E_{\alpha_1}^{(n-1)} - E_{-\alpha_1}^{(n+1)}\right) - \left(E_{\alpha_3}^{(n-1)} - E_{-\alpha_3}^{(n+1)}\right), \\
M_5^{(n)} &= (-\alpha_1 + \alpha_3) \cdot H^{(n)}, \\
M_6^{(n)} &= -\left(E_{\alpha_1}^{(n-1)} - E_{-\alpha_1}^{(n+1)}\right) + \left(E_{\alpha_3}^{(n-1)} - E_{-\alpha_3}^{(n+1)}\right), \\
M_7^{(n)} &= (-\alpha_5 + \alpha_7) \cdot H^{(n)}, \\
M_8^{(n)} &= -\left(E_{\alpha_7}^{(n+1)} - E_{-\alpha_7}^{(n-1)}\right) + \left(E_{\alpha_5}^{(n+1)} - E_{-\alpha_5}^{(n-1)}\right), \\
(M_9^\pm)^{(n)} &= (E_{\alpha_1+\alpha_2}^{(n)} \pm E_{-\alpha_1-\alpha_2}^{(n)}) - (E_{\alpha_2+\alpha_3}^{(n)} \pm E_{-\alpha_2-\alpha_3}^{(n)}), \\
(M_{10}^\pm)^{(n)} &= (E_{\alpha_5+\alpha_6}^{(n)} \pm E_{-\alpha_5-\alpha_6}^{(n)}) - (E_{\alpha_6+\alpha_7}^{(n)} \pm E_{-\alpha_6-\alpha_7}^{(n)}), \\
(M_{11}^\pm)^{(n)} &= (E_{\alpha_1+\alpha_2+\alpha_3}^{(n-1)} \pm E_{-\alpha_1-\alpha_2-\alpha_3}^{(n+1)}) - (E_{\alpha_2}^{(n+1)} \pm E_{-\alpha_2}^{(n-1)}), \\
(M_{12}^\pm)^{(n)} &= (E_{\alpha_5+\alpha_6+\alpha_7}^{(n+1)} \pm E_{-\alpha_5-\alpha_6-\alpha_7}^{(n-1)}) - (E_{\alpha_5}^{(n+1)} \pm E_{-\alpha_5}^{(n-1)}), \tag{A.4}
\end{aligned}$$

A.2 Reduction of $\widehat{sl}(4|4)$

Reduction is performed in two steps.

The first step consists of an algebraic reduction of $sl(4|4)$ algebra leading to a subalgebra of with Grassmannian generators written in the form (with $i = 1, \dots, 8$):

$$\begin{aligned}
R(\eta_i, \zeta_i) &= (\eta_1 E_{\alpha_1+\dots+\alpha_4} + \zeta_1 E_{-\alpha_1-\dots-\alpha_4}) + (\eta_2 E_{\alpha_1+\dots+\alpha_5} + \zeta_2 E_{-\alpha_1-\dots-\alpha_5}) \\
&+ (\eta_3 E_{\alpha_2+\dots+\alpha_4} + \zeta_3 E_{-\alpha_2-\dots-\alpha_4}) + (\eta_4 E_{\alpha_2+\dots+\alpha_5} + \zeta_4 E_{-\alpha_2-\dots-\alpha_5}) \\
&+ (\eta_5 E_{\alpha_3+\dots+\alpha_6} + \zeta_5 E_{-\alpha_3-\dots-\alpha_6}) + (\eta_6 E_{\alpha_3+\dots+\alpha_7} + \zeta_6 E_{-\alpha_3-\dots-\alpha_7}) \\
&+ (\eta_7 E_{\alpha_4+\dots+\alpha_6} + \zeta_7 E_{-\alpha_4-\dots-\alpha_6}) + (\eta_8 E_{\alpha_4+\dots+\alpha_7} + \zeta_8 E_{-\alpha_4-\dots-\alpha_7}), \tag{A.5}
\end{aligned}$$

Denote

$$\begin{aligned}
F_1 &= R(\eta_i = 1, \zeta_i = 1), \\
F_2 &= R(\eta_i = -1, \zeta_i = 1), \\
F_3 &= R(\eta_1 = \eta_4 = \eta_5 = \eta_8 = \zeta_1 = \zeta_4 = \zeta_5 = \zeta_8 = -1, \text{ all other } \eta_i = \zeta_i = 1), \\
F_4 &= R(\eta_2 = \eta_3 = \eta_6 = \eta_7 = \zeta_1 = \zeta_4 = \zeta_5 = \zeta_8 = -1, \text{ all other } \eta_i = \zeta_i = 1), \\
F_5 &= R(\eta_1 = \eta_4 = \eta_6 = \eta_7 = \zeta_1 = \zeta_4 = \zeta_6 = \zeta_7 = -1, \text{ all other } \eta_i = \zeta_i = 1), \\
F_6 &= R(\eta_2 = \eta_3 = \eta_5 = \eta_8 = \zeta_1 = \zeta_4 = \zeta_6 = \zeta_7 = -1, \text{ all other } \eta_i = \zeta_i = 1), \\
F_7 &= R(\eta_5 = \eta_6 = \eta_7 = \eta_8 = \zeta_5 = \zeta_6 = \zeta_7 = \zeta_8 = -1, \text{ all other } \eta_i = \zeta_i = 1), \\
F_8 &= R(\eta_1 = \eta_2 = \eta_3 = \eta_4 = \zeta_5 = \zeta_6 = \zeta_7 = \zeta_8 = -1, \text{ all other } \eta_i = \zeta_i = 1), \tag{A.6}
\end{aligned}$$

and

$$\begin{aligned}
G_1 &= R(\eta_2 = \eta_4 = \eta_6 = \eta_8 = \zeta_2 = \zeta_4 = \zeta_6 = \zeta_8 = -1, \text{ all other } \eta_i = \zeta_i = 1), \\
G_2 &= R(\eta_1 = \eta_3 = \eta_5 = \eta_7 = \zeta_2 = \zeta_4 = \zeta_6 = \zeta_8 = -1, \text{ all other } \eta_i = \zeta_i = 1), \\
G_3 &= R(\eta_1 = \eta_2 = \eta_5 = \eta_6 = \zeta_1 = \zeta_2 = \zeta_5 = \zeta_6 = -1, \text{ all other } \eta_i = \zeta_i = 1), \\
G_4 &= R(\eta_3 = \eta_4 = \eta_7 = \eta_8 = \zeta_1 = \zeta_2 = \zeta_5 = \zeta_6 = -1, \text{ all other } \eta_i = \zeta_i = 1), \\
G_5 &= R(\eta_1 = \eta_2 = \eta_7 = \eta_8 = \zeta_1 = \zeta_2 = \zeta_7 = \zeta_8 = -1, \text{ all other } \eta_i = \zeta_i = 1), \\
G_6 &= R(\eta_3 = \eta_4 = \eta_5 = \eta_6 = \zeta_1 = \zeta_2 = \zeta_7 = \zeta_8 = -1, \text{ all other } \eta_i = \zeta_i = 1), \\
G_7 &= R(\eta_2 = \eta_4 = \eta_5 = \eta_7 = \zeta_2 = \zeta_4 = \zeta_5 = \zeta_7 = -1, \text{ all other } \eta_i = \zeta_i = 1), \\
G_8 &= R(\eta_1 = \eta_3 = \eta_6 = \eta_8 = \zeta_2 = \zeta_4 = \zeta_5 = \zeta_7 = -1, \text{ all other } \eta_i = \zeta_i = 1), \quad (\text{A.7})
\end{aligned}$$

It is possible to show that the given generators $F_i, G_i, M_i, \quad i = 1, \dots, 8, K_a, K_{a+3}, a = 1, 2, 3$ and I of $sl(4|4)$ closes into a subalgebra using the following commutation relations

$$\begin{aligned}
[H_i, H_j] &= 0, \\
[H_i, E_a] &= \alpha^i E_a, \\
[E_\alpha, E_\beta] &= \begin{cases} \epsilon(\alpha, \beta) E_{\alpha+\beta}, & \alpha + \beta = \text{root}, \\ \alpha \cdot H, & \alpha + \beta = 0, \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

where we $\epsilon(\alpha, \beta)$ is the structure constant.

As the next second step we introduce a loop structure within such subalgebra by multiplying each generator by λ^r , where $r \in \mathbb{Z}$ for the bosonic and $r \in \mathbb{Z} + \frac{1}{2}$ fermionic generators, respectively. In the present paper we shall be using the following loop reduction of the algebra defined by

$$\begin{aligned}
&\{M_1^{(r)}, M_3^{(r)}, M_5^{(r)}, M_7^{(r)}\}, \quad r = 2n, \\
&\{M_2^{(r)}, M_4^{(r)}, M_6^{(r)}, M_8^{(r)}, K_1^{(r)}, K_2^{(r)}, K_3^{(r)}, K_6^{(r)}, K_7^{(r)}, K_8^{(r)}, I^{(r)}\} \quad r = 2n + 1, \\
&\{G_1^{(r)}, G_3^{(r)}, G_5^{(r)}, G_7^{(r)}, F_2^{(r)}, F_4^{(r)}, F_6^{(r)}, F_8^{(r)}\}, \quad r = 2n + \frac{1}{2}, \\
&\{G_2^{(r)}, G_4^{(r)}, G_6^{(r)}, G_8^{(r)}, F_1^{(r)}, F_3^{(r)}, F_5^{(r)}, F_7^{(r)}\}, \quad r = 2n + \frac{3}{2}, \quad (\text{A.8})
\end{aligned}$$

where $n \in \mathbb{Z}$. The first four generators, namely M_1, M_3, M_5, M_7 defined in equation (A.4) involve the following Cartan subalgebra elements

$$\alpha_1 \cdot H, \quad \alpha_3 \cdot H, \quad \alpha_5 \cdot H, \quad \alpha_7 \cdot H \quad (\text{A.9})$$

while the kernel generators K_3, K_7 and I defined in equation (A.3) involve the other three Cartan subalgebra generators

$$\alpha_2 \cdot H, \quad \alpha_4 \cdot H, \quad \alpha_6 \cdot H. \quad (\text{A.10})$$

The other generators $M_2, M_4, M_6, M_8, K_1, K_2, K_6, K_8$ give raise to the following step operators

$$E_{\pm\alpha_1}, \quad E_{\pm\alpha_3}, \quad E_{\pm\alpha_5}, \quad E_{\pm\alpha_7}. \quad (\text{A.11})$$

Finally, the 16 generators $G_i, F_i, i = 1, \dots, 8$ in (A.6) and (A.7) are linear combinations of step operators :

$$\begin{aligned} E_{\pm(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}, E_{\pm(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5)}, E_{\pm(\alpha_2+\alpha_3+\alpha_4)}, E_{\pm(\alpha_2+\alpha_3+\alpha_4+\alpha_5)}, \\ E_{\pm(\alpha_3+\alpha_4+\alpha_5+\alpha_6)}, E_{\pm(\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7)}, E_{\pm(\alpha_4+\alpha_5+\alpha_6)}, E_{\pm(\alpha_4+\alpha_5+\alpha_6+\alpha_7)}. \end{aligned} \quad (\text{A.12})$$

The key point we want to make here is that we may decompose the subalgebra in (A.8) into two commuting $sl(2, 2)$ sectors shown below.

A.3 $\beta - sl(2, 2)$

$$\begin{aligned} E_{\pm\beta_1} &= E_{\pm\alpha_1}, \\ E_{\pm\beta_2} &= E_{\pm(\alpha_2+\alpha_3+\alpha_4)}, \\ E_{\pm\beta_3} &= E_{\pm\alpha_5}, \\ E_{\pm(\beta_1+\beta_2)} &= E_{\pm(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}, \\ E_{\pm(\beta_2+\beta_3)} &= E_{\pm(\alpha_2+\alpha_3+\alpha_4+\alpha_5)}, \\ E_{\pm(\beta_1+\beta_2+\beta_3)} &= E_{\pm(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5)}, \\ \beta_1 \cdot H &= \alpha_1 \cdot H, \\ \beta_3 \cdot H &= \alpha_5 \cdot H, \\ \beta_2 \cdot H &= (\alpha_2 + \alpha_3 + \alpha_4) \cdot H \end{aligned} \quad (\text{A.13})$$

An identity element (in the sense that commutes with all generators in (A.13)) within such subalgebra is given by $I_\beta = (\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5) \cdot H$. The second $sl(2, 2)$ sector is generated by

A.4 $\gamma - sl(2, 2)$

$$\begin{aligned} E_{\pm\gamma_1} &= E_{\pm\alpha_3}, \\ E_{\pm\gamma_2} &= E_{\pm(\alpha_4+\alpha_5+\alpha_6)}, \\ E_{\pm\gamma_3} &= E_{\pm\alpha_7}, \\ E_{\pm(\gamma_1+\gamma_2)} &= E_{\pm(\alpha_1+\alpha_2+\alpha_3+\alpha_4)}, \\ E_{\pm(\gamma_2+\gamma_3)} &= E_{\pm(\alpha_4+\alpha_5+\alpha_6+\alpha_7)}, \\ E_{\pm(\gamma_1+\gamma_2+\gamma_3)} &= E_{\pm(\alpha_3+\alpha_4+\alpha_5+\alpha_6+\alpha_7)}, \\ \gamma_1 \cdot H &= \alpha_3 \cdot H, \\ \gamma_3 \cdot H &= \alpha_7 \cdot H, \\ \gamma_2 \cdot H &= (\alpha_4 + \alpha_5 + \alpha_6) \cdot H \end{aligned} \quad (\text{A.14})$$

An identity element (in the sense that it commutes with all generators in (A.14)) within such subalgebra is given by $I_\gamma = (\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7) \cdot H$.

Notice that the total identity (i.e. identity of $sl(4, 4)$) is given as $I = I_\beta + I_\gamma$, and therefore is not linearly independent of the Cartan subalgebra generators in (A.13) and

(A.14). However the subalgebra described by generators in (A.9), (A.10), (A.11) and (A.12) have 31 generators. Each $sl(2, 2)$ subalgebra (A.13) and (A.14) have 15 generators. There is one extra generator (which we may choose to be $\alpha_4 \cdot H$ for symmetry) that does not commute with neither $sl(2, 2)$ in (A.13) or in (A.14). The relevant subalgebra consists therefore of a semi direct product of $U(1)$ with $sl(2, 2) \otimes sl(2, 2)$.

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